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Quantum Mechanics: General uncertainty relation

Uncertainty in observables

We all start out by learning about the Heisenberg uncertainty relation between position and momentum

$$\Delta x \Delta p \geq \frac{\hbar}{2},$$

where $\Delta x, \Delta p$ is the uncertainty in position and momentum, respectively. We grew up hearing, the more precisely you try to measure position, the more you disturb the momentum, and vice-versa. One gets a feeling that it has something to do with the process of measurement. In the following we will learn that this is not really so. We know that in quantum mechanics all observable cannot have a precise value (eigenvalue) at the same time. That is the origin of the uncertainty. In other words, it doesn't have to do with the process of measurement, the structure of quantum mechanics is such that there is a limit to the preciseness with which the value of an observable can be defined, *given a particular state*.

We know that for an observable $\hat{A}$, its value is precisely defined if the system is in one of the eigenstates of $\hat{A}$. For example, if the state is $|a_k\rangle$, such that $\hat{A}|a_k\rangle = \alpha_k|a_k\rangle$, the value of the observable is precisely α_k. There is no uncertainty in that. However, if the state is $|\psi\rangle$, which is not an eigenstate of $\hat{A}$, how does one define the value of $\hat{A}$? We have learnt that in such a situation one can only talk of the *expection value* of the observable

$$\langle\hat{A}\rangle = \langle\psi|\hat{A}|\psi\rangle.$$

We learnt that the expection value can be understood as an average value, average over many measurements. In statstics, when there is a statistical variable, we talk of its average. But to know how far it can deviate from the average, we talk of *variance* or *standard deviation*. In quantum mechanics too, we want to know how much the value of an observable deviate from its expectation value. A measure like standard deviation is what contitutes the uncertainty. We define the uncertainty exactly the way standard deviation is defined:

$$\Delta A \equiv \sqrt{\langle(\hat{A} - \langle\hat{A}\rangle)^2\rangle},$$

where the angular brackets denote expectation value. For another observable $\hat{B}$, the uncertainty is given by

$$\Delta B \equiv \sqrt{\langle(\hat{B} - \langle\hat{B}\rangle)^2\rangle}.$$

As the expectation value depends on the state of the system, so does uncertainty. So the uncertainty of an observable could be more or less, depending on the state of the system. Needless to say that since we talking about *observables*, both the operators are Hermitian.

For calculational convenience, we define modified operators as

$$\hat{A}' = \hat{A} - \langle\hat{A}\rangle, \quad \hat{B}' = \hat{B} - \langle\hat{B}\rangle.$$

There operators acting on the state $|\psi\rangle$, modify it as

$$|\phi_1\rangle = \hat{A}'|\psi\rangle, \quad |\phi_2\rangle = \hat{B}'|\psi\rangle.$$

Now, for any two (unnormalized) states $|\phi_1\rangle, |\phi_2\rangle$, the *Schwarz inequality* states that

$$\langle\phi_1|\phi_1\rangle\langle\phi_2|\phi_2\rangle \geq |\langle\phi_1|\phi_2\rangle|^2,$$

so it should hold for our specifically defined $|\phi_1\rangle, |\phi_2\rangle$ too. We notice that

$$\langle\phi_1|\phi_1\rangle = \langle\psi|\hat{A}'^2|\psi\rangle = \langle\psi|(\hat{A} - \langle\hat{A}\rangle)^2|\psi\rangle = \Delta A^2,$$

$$\langle\phi_2|\phi_2\rangle = \langle\psi|\hat{B}'^2|\psi\rangle = \langle\psi|(\hat{B} - \langle\hat{B}\rangle)^2|\psi\rangle = \Delta B^2.$$

Thus the Schwarz inequality assumes the form

$$\Delta A^2 \Delta B^2 \geq |\langle\psi|\hat{A}'\hat{B}'|\psi\rangle|^2$$

We can rewrite $\hat{A}'\hat{B}'$ as

$$\hat{A}'\hat{B}' = \tfrac{1}{2}[\hat{A}', \hat{B}'] + \tfrac{1}{2}\{\hat{A}', \hat{B}'\},$$

where $\{\ ,\ \}$ represents the *anticommutator*. The Schwar
inequality is then

$$\Delta A^2 \Delta B^2 \geq \tfrac{1}{4}|\langle\psi|[\hat{A}', \hat{B}']|\psi\rangle + \langle\psi|\{\hat{A}', \hat{B}'\}|\psi\rangle|^2.$$

Now we can use an interesting property of commutator and anticommutator. An anticommutator is a Hermitian operator, and so its expectation value is always real. A commutator is an anti-Hermitian operator, and so its expectation value is always purely imaginary. So, on the RHS we have the absolute value of the sum of a real and imaginary number. We know that

$$|c_1 + ic_2|^2 = c_1^2 + c_2^2,$$

which leads us to

$$\Delta A^2 \Delta B^2 \geq \tfrac{1}{4}|\langle\psi|[\hat{A}', \hat{B}']|\psi\rangle|^2 + \tfrac{1}{4}|\langle\psi|\{\hat{A}', \hat{B}'\}|\psi\rangle|^2.$$

The LHS is $\geq$ the sum of two terms on the RHS, so obviously it will also be $\geq$ *one* of those terms. We discard the anticommutator term, and write the inequality as

$$\Delta A^2 \Delta B^2 \geq \tfrac{1}{4}|\langle\psi|[\hat{A}', \hat{B}']|\psi\rangle|^2.$$

Now $[\hat{A}', \hat{B}'] = [\hat{A}, \hat{B}]$, which then simplifies the above to

$$\boxed{\Delta A^2 \Delta B^2 \geq \tfrac{1}{4}|\langle\psi|[\hat{A}, \hat{B}]|\psi\rangle|^2}.$$

Taking square root of both sides, we get

$$\boxed{\Delta A \Delta B \geq \tfrac{1}{2}|\langle\psi|[\hat{A}, \hat{B}]|\psi\rangle|}. \qquad (1)$$

This is our *general uncertainty relation* between any two observables. It can be verified that for $\hat{x}$ and $\hat{p}$, it gives

$$\Delta x \Delta p \geq \tfrac{\hbar}{2}.$$

It is noteworthy that if two operators $\hat{P}, \hat{Q}$ commute, then

$$\Delta P \Delta Q \geq 0,$$

which implies that there exist states in which the uncertainties of both $\hat{P}$ and $\hat{Q}$ are zero. Those are precisely the common eigenstates of $\hat{P}$ and $\hat{Q}$, which exist since they are commuting operators.

Limitations of the general uncertainty relation

The general uncertainty relation (1) is at the very core of quantum mechanics. The derivation shows that it cannot be violated. It is just a result of the properties of states and operators. However, it suffers from some limitations which we will discuss here. The idea of the uncertainty relation is that if the uncertainty of one observable is known, the relation should provide a bound on the uncertainty of the other observable. For most cases the general uncertainty relation does the job. However, consider a state such that it is an eigenstate of the observable $\hat{A}$. In this case $\Delta A = 0$. The LHS of (1) becomes zero, and there is no way to estimate the bound on ΔB.

Another problem arises in the situation where $[\hat{A}, \hat{B}] = i\hat{C}$, and the state is such that $\langle\hat{C}\rangle = 0$. This can happen in very normal cases like that of angular momentum operators $\hat{L}_x, \hat{L}_y$. In such a case the relation (1) reduces to

$$\Delta A \Delta B \geq 0.$$

Now, $\Delta A \Delta B$ being non-negative measures, there product is anyway bounded from below by zero. So the uncertainty relation gives a *trivial bound* in this case. It can be shown that there are states where $\langle\hat{C}\rangle = 0$, but neither ΔA, nor ΔB is zero. So there should exist bounds but the general uncertainty relation is unable to provide one.

To address these problems, a new uncertainty relation was formulated by *Lorenzo Maccone* and *Arun Pati* in

2014. It is known as the **Maccone–Pati uncertainty relation**, and has the following form:

$$\Delta A^2 + \Delta B^2 \geq i\langle\psi|[\hat{A},\hat{B}]|\psi\rangle + |\langle\psi|\hat{A} + i\hat{B}|\psi^{\perp}\rangle|^2,$$

where $|\psi^{\perp}\rangle$ is an arbitrary state orthogonal to the system state $|\psi\rangle$, and the sign should be chosen so that $i\langle[\hat{A},\hat{B}]\rangle$ (a real quantity) is positive.