Electrodynamics

Lecture 3.4
Green's Functions

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§ 1

Superposition Principle

The static electric field due to a charge density $\rho(\mathbf{r})$ is governed by the Poisson eqution

$$\nabla^2 \mathbf{E} = -\frac{\rho}{\epsilon_0}$$

This equation is linear in $\mathbf{E}$. If the charge density can be written as a sum (or superposition) of two densities $\rho(\mathbf{r}) = \rho_1(\mathbf{r}) + \rho_2(\mathbf{r})$ with electric fields $\mathbf{E_1}$ and $\mathbf{E_2}$ respectively then using

$$\nabla^2 \mathbf{E_1} = -\frac{\rho_1(\mathbf{r})}{\epsilon_0}, \qquad \text{and} \qquad \nabla^2 \mathbf{E_2} = -\frac{\rho_2(\mathbf{r})}{\epsilon_0}$$

we can infer that

$$\nabla^2 (\mathbf{E_1} + \mathbf{E_2}) = -\frac{\rho_1(\mathbf{r}) + \rho_2(\mathbf{r})}{\epsilon_0}.$$

This shows that the field solution for a sum of charge distributions can be obtained by simply adding the solutions for individual distributions.

We can continue the process of dividing the charge distributions indefinitely till the limit of point charges. A point charge density is given by a Dirac delta function. Therefore if we can solve the problem of the field of a point charge, we can solve it for any other charge distribution.

The solutions of linear partial differential equations (PDE) of this kind where the "right hand side" of the equation corresponding to the *source* of the field is a delta function are called **Green's functions** or **fundamental solutions**.

§ 2

Three dimensional Dirac-delta

First we define the three-dimensional Dirac delta function $\delta^3(\mathbf{r})$ as

$$\begin{aligned}\delta^3(\mathbf{r}) &= \delta(x)\delta(y)\delta(z) \\ &= \left(\frac{1}{(2\pi)}\int e^{ik_x x}\,dk_x\right)(\text{similar integral for } k_y)\,(\text{for } k_z) \\ &= \frac{1}{(2\pi)^3}\int e^{i\mathbf{k}.\mathbf{r}}\,d^3\mathbf{k}\end{aligned}$$

The function has value zero at all points except the origin $\mathbf{r} = 0$, and is such that for any smooth function ϕ of $\mathbf{r}$, the integral

$$\int d^3\mathbf{r}\delta(\mathbf{r})\phi(\mathbf{r}) = \phi(0)$$

In particular we can write (choosing for ϕ a smooth function which is equal to unity around the origin)

$$\int_V d^3\mathbf{r}\delta(\mathbf{r}) = 1$$

where V is a region including the origin. The integral is zero when the region V excludes the origin $\mathbf{r} = 0$. This shows, for example, that $q\delta(\mathbf{r})$ represents a charge density of a point particle located at the origin with charge q.

§ 3

Green's function for the Poisson Equation

The method of Green's function involves solving the equation

$$\nabla^2 G(\mathbf{r}-\mathbf{r}') = \delta^3(\mathbf{r}-\mathbf{r}')$$

Suppose we are able to solve this equation. Then the solution to our original Poisson equation can be written as

$$\Phi(\mathbf{r}) = -\frac{1}{\epsilon_0}\int d^3\mathbf{r}' G(\mathbf{r}-\mathbf{r}')\rho(\mathbf{r}')$$

because

$$\begin{aligned}
\nabla^2 \int d^3\mathbf{r}' G(\mathbf{r}-\mathbf{r}')\rho(\mathbf{r}') &= \int d^3\mathbf{r}' \nabla^2\, G(\mathbf{r}-\mathbf{r}')\rho(\mathbf{r}') \\
&= \int d^3\mathbf{r}' \delta^3(\mathbf{r}-\mathbf{r}')\rho(\mathbf{r}') \\
&= \rho(\mathbf{r})
\end{aligned}$$

Before the calculation of the Green's function we give the answer and verify that it actually is the solution. The solution is

$$\nabla^2\left(-\frac{1}{4\pi r}\right) = \delta^3(\mathbf{r})$$

and you should commit it to memory. This is a very important formula.

By expressing ∇^2 in polar coordinates

$$\nabla^2 = \frac{1}{r}\frac{\partial^2}{\partial r^2}r + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

we see that $\nabla^2(1/r) = 0$ for any $r \neq 0$. Therefore we must check

$$\int_V d^3\mathbf{r}\, \nabla^2 \left(-\frac{1}{4\pi r}\right) = 1$$

when the volume V includes the origin.

Let us take the region V to be a sphere of radius a centered at the origin. Then using the Gauss theorem

$$\int_V d^3\mathbf{r}\, \nabla. \left[\nabla \left(-\frac{1}{4\pi r}\right)\right] = -\frac{1}{4\pi} \int_S \nabla \left(\frac{1}{r}\right) .\mathbf{n}\, dS$$

where $\mathbf{n}$ is the unit radial vector on spherical surface S. Using the fact that at the boundary S of V where $r = a \neq 0$,

$$\nabla \left(\frac{1}{r}\right) = -\frac{\mathbf{r}}{r^3} = -\frac{\mathbf{n}}{r^2} = -\frac{\mathbf{n}}{a^2}$$

we get

$$\frac{1}{4\pi a^2} \int_S \mathbf{n}.\mathbf{n}\, dS = \frac{1}{4\pi a^2} \int_S dS = 1$$

which proves the result.

§ 4

Calculation of the Green's Function

The green's functions are solutions of PDE's which involve generalized function like the Dirac delta. Therefore they are generalized functions themselves.

Normally, one begins the calculation of Green's function by assuming a Fourier transform

$$G(\mathbf{r}) = \frac{1}{(2\pi)^3} \int d^3\mathbf{k}\ \widetilde{G}\,(\mathbf{k})\, e^{i\mathbf{k}.\mathbf{r}}$$

Then, applying ∇^2

$$\nabla^2 G(\mathbf{r}) = \frac{1}{(2\pi)^3} \int d^3\mathbf{k}\, (-\mathbf{k}^2)\, \tilde{G}\, (\mathbf{k})\, e^{i\mathbf{k}.\mathbf{r}}$$

and equating it to the fourier expansion of the delta function

$$\delta(\mathbf{r}) = \frac{1}{(2\pi)^3} \int d^3\mathbf{k}\, e^{i\mathbf{k}.\mathbf{r}}$$

we get

$$\tilde{G}\, (\mathbf{k}) = -\frac{1}{\mathbf{k}^2}$$

So the Greens function is

$$G(\mathbf{r}) = \frac{-1}{(2\pi)^3} \int d^3\mathbf{k}\, \frac{1}{\mathbf{k}^2}\, e^{i\mathbf{k}.\mathbf{r}}$$

provided we can integrate the right hand side. But we cannot. The right hand side is singular because of the $1/k^2$ factor.

By a method which should, by now, be familiar to you, we re-interpret the integral by introducing a small positive number ϵ and equating it as the limit $\epsilon \to 0$ of

$$G_\epsilon(\mathbf{r}) \equiv \frac{-1}{(2\pi)^3} \int d^3\mathbf{k}\, \frac{1}{\mathbf{k}^2 + \epsilon^2}\, e^{i\mathbf{k}.\mathbf{r}}$$

There are a number of 'tricks' used at this stage :

1. The integral on the right hand side in independent of what directions are chosen for the axes in the k-space. For the fixed value of $\mathbf{r}$ we chose the direction of k_z axis along $\mathbf{r}$. Then calling the polar coordinates in the k-space by k, θ, ϕ we get $\mathbf{k}.\mathbf{r} = kr\cos\theta$,

2. Integrating ϕ gives a factor of 2π.

3. Calling $\cos\theta = u$

$$\begin{aligned} G_\epsilon(\mathbf{r}) &= \frac{-1}{(2\pi)^2}\int_0^\infty k^2\,dk\int_{-1}^1 du\,\frac{e^{ikru}}{k^2+\epsilon^2} \\ &= \frac{-1}{ir(2\pi)^2}\int_0^\infty k\,dk\,\frac{e^{ikr}-e^{-ikr}}{k^2+\epsilon^2} \\ &= \frac{-1}{ir(2\pi)^2}\int_{-\infty}^\infty k\,dk\,\frac{e^{ikr}}{k^2+\epsilon^2} \end{aligned}$$

where the two integrals in k over half range $(0,\infty)$ are combined into a single integral over $(-\infty,+\infty)$.

4. The integral over k can be interpreted as a complex k-integration along a closed contour running along the real axis from $-\infty$ to ∞ and along an infinite semicircle in the upper half plane. Why upper half plane? Because on this part of the contour the integrand is zero : $\exp(i(k+i\eta)r) = \exp(-\eta r + ikr) \to 0$ for large η. (In the lower half plane a similar contour will spell disaster : the integrand blows up exponentially!)

5. There is one pole in the upper half plane, at $k = i\epsilon$,

$$k\frac{e^{ikr}}{k^2+\epsilon^2} = \frac{e^{ikr}}{2}\left[\frac{1}{k+i\epsilon}+\frac{1}{k-i\epsilon}\right]$$

and the "residue theorem" of complex integration gives

$$\begin{aligned} G_\epsilon(\mathbf{r}) &= \frac{-1}{ir(2\pi)^2}\int_{-\infty}^\infty k\,dk\,\frac{e^{ikr}}{k^2+\epsilon^2} \\ &= \frac{-1}{ir(2\pi)^2}(2\pi i)\frac{e^{-\epsilon r}}{2} \\ &= -\frac{e^{-\epsilon r}}{4\pi r} \end{aligned}$$

Remark 1 The result given above is itself of great importance. We have not used the smallness of ϵ anywhere. Thus the Green's function $G_m$of the "static Klein-Gordon equation"

$$(\nabla^2 - m^2)G(\mathbf{r}) = \delta(\mathbf{r})$$

is

$$G_m(\mathbf{r}) = -\frac{e^{-mr}}{4\pi r}.$$

It is called the "Yukawa potential".

The Green's function for the Poisson equation is, finally,

$$\begin{aligned} G(\mathbf{r}) &= \lim_{\epsilon \to 0} G_\epsilon(\mathbf{r}) \\ &= \lim_{\epsilon \to 0} -\frac{e^{-\epsilon r}}{4\pi r} \\ &= -\frac{1}{4\pi r} \end{aligned}$$

§ 5

Green's function for the Helmholtz equation

The Helmholtz equation is

$$(\nabla^2 + \kappa^2)\phi = \text{ source term,} \qquad \kappa = \text{ a real number.}$$

Its Green's function satisfies

$$(\nabla^2 + \kappa^2)G(\mathbf{r}) = \delta(\mathbf{r})$$

The Fourier transform $\widetilde{G}\,(\mathbf{k})$ of $G(\mathbf{r})$ is

$$G(\mathbf{r}) = \frac{1}{(2\pi)^3} \int d^3\mathbf{k}\ \widetilde{G}\,(\mathbf{k})\, e^{i\mathbf{k}.\mathbf{r}}$$

where

$$\widetilde{G} = -\frac{1}{|\mathbf{k}|^2 - \kappa^2}$$

The integral is problematic because the integrand is singular at $k = |\mathbf{k}| = \pm\kappa$.

$$G(\mathbf{r}) = -\frac{1}{(2\pi)^3} \int d^3\mathbf{k} \, \frac{1}{k^2 - \kappa^2} \, e^{i\mathbf{k}.\mathbf{r}}$$

We trace the same steps as before : as far as possible.

1. Choose the 'z-axis' in k-space along $\mathbf{r}$ so that $\mathbf{k}.\mathbf{r} = kr\cos\theta$.
2. $d^3\mathbf{r} = k^2 dk d(\cos\theta) d\phi$ with ranges $k = (0, \infty), \cos\theta = (-1, 1), \phi = (0, 2\pi)$.
3. $d\phi$ integration gives a factor of 2π.
4. $d(\cos\theta)$ integration of $\exp(ikr\cos\theta)$ gives
$$\frac{1}{ikr}(e^{ikr} - e^{-ikr})$$
5. Combine the two exponentials (with $k = 0, \infty$) into a single integral of $\exp(ikr)$ running from $-\infty$ to ∞.

Thus we have

$$\begin{aligned} G(\mathbf{r}) &= -\frac{1}{4ir\pi^2} \int_{-\infty}^{\infty} \frac{e^{ikr} k dk}{k^2 - \kappa^2} \\ &= -\frac{1}{8ir\pi^2} \left(\int_{-\infty}^{\infty} \frac{e^{ikr} dk}{k - \kappa} + \int_{-\infty}^{\infty} \frac{e^{ikr} dk}{k + \kappa} \right) \end{aligned}$$

Again, we encounter a function defined through a singular integral. The cure is to redefine the integral as a generalized function with an infinitesimal parameter ϵ so that the original function is obtained in the limit of $\epsilon \to 0$.

The two integrals can be redefined by changing $\kappa \to \kappa \pm i\epsilon$.

$$G_{\pm}(\mathbf{r}) = -\frac{1}{8ir\pi^2}\left(\int_{-\infty}^{\infty}\frac{e^{ikr}dk}{k-\kappa\mp i\epsilon} + \int_{-\infty}^{\infty}\frac{e^{ikr}dk}{k+\kappa\pm i\epsilon}\right)$$

As $r > 0$, the integral can only be "closed in the uhp" (upper half plane of k) because then the contribution of the semi-circular contour is zero and the real line integral from $-\infty$ to ∞ can be converted into a contour integral. The pole in the upper half plane for G_+ is at $k = \kappa + i\epsilon$ integral can be calculated by residue theorem :

$$G_+(\mathbf{r}) = -\frac{1}{8ir\pi^2}(2\pi i)e^{i\kappa r} = -\frac{e^{i\kappa r}}{4\pi r}.$$

Similarly, for G_- the pole falling in uhp is at $-\kappa + i\epsilon$ and contributes

$$G_-(\mathbf{r}) = -\frac{e^{-i\kappa r}}{4\pi r}.$$

These two solutions $G_{\pm}$ are called the "outgoing" and "incoming" wave solutions.

Remark 2 Why is there one solution of the Poisson equation and two for the Helmholtz? Actually, for any inhomogeneous equation the general solution is always of the form

general solution = a particular solution of the inhomogeneous equation
+ a general solution of the homogeneous equation.

We can think of either G_+ or G_- as the particular solution, then the other one is obtained by adding a solution of the homogeneous solution to it. For example,

$$G_- = G_+ + (G_- - G_+),$$

and the difference $G_- - G_+$ satisfies the homogeneous equation.

§ 6

Green's function for the Wave equation

Helmholtz equation simplifies the derivation of the Green's function for the wave equation. The wave equation is

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi = \text{ source term,}$$

and its Green's function $G(t, \mathbf{r})$ satisfies

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G = \delta^3(\mathbf{r})\delta(t).$$

Assume the Fourier transform of four variables,

$$G(\mathbf{r}, t) = \frac{1}{(2\pi)^4}\int d^3\mathbf{k}\int d\omega\ \widetilde{G}\ (\mathbf{k}, \omega)\, e^{i\mathbf{k}.\mathbf{r}-i\omega t}.$$

The minus sign in the Fourier exponential for time variable t is conventional.

(It has a deep reason though. That has something to do with the structure of space-time being Minkowski and not Euclidean. Or, with the energy of a physical system being bounded from below while there is no such restriction on the momenta. Or, with why in quantum mechanics the energy is $i\hbar\partial/\partial t$ but momenta are $-i\hbar\nabla$. But we do not go into that here because that is irrelevant. If it makes you feel better, you can substitute $-\omega$ for ω everywhere with absolutely no change in anything but the notation.)

We solve this problem in two steps. First, we define the intermediate Green's function

$$g(\mathbf{r}, \omega) = \frac{1}{(2\pi)^3}\int d^3\mathbf{k}\ \widetilde{G}\ (\mathbf{k}, \omega)\, e^{i\mathbf{k}.\mathbf{r}}.$$

The Green's function is thus determined by,

$$G(\mathbf{r},t) = \frac{1}{(2\pi)} \int d\omega\, g(\mathbf{r},\omega)\, e^{-i\omega t},$$

and the function g satisfies the Helmholts equation

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right) g(\mathbf{r},\omega) \equiv \left(\nabla^2 + \kappa^2\right) g(\mathbf{r},\omega) = \delta(\mathbf{r})$$

where $\omega = \kappa c$. The tw0 solutions for the Helmholtz equations are

$$g_\pm = -\frac{e^{\pm i\kappa r}}{4\pi r},$$

which gives

$$\begin{aligned} G_\pm(\mathbf{r},t) &= \frac{1}{(2\pi)} \int d\omega\, g_\pm(\mathbf{r},\omega)\, e^{-i\omega t} \\ &= -\frac{1}{4\pi r}\frac{1}{(2\pi)} \int d\omega\, e^{\pm i\kappa r - i\omega t} \\ &= -\frac{1}{4\pi r}\frac{1}{(2\pi)} \int d\omega\, e^{\pm i\omega r/c - i\omega t} \\ &= -\frac{1}{4\pi r}\delta\left(t \mp \frac{r}{c}\right) \end{aligned}$$

§ 7

Green's function for the heat equation

[**Not really required for the Electrodynamics course.**] **In this case we have a function on both r and t.**

The equation to be solved is

$$\frac{\partial G(\mathbf{r},t)}{\partial t} = D\nabla^2\, G(\mathbf{r},t) + \delta^3(\mathbf{r})\delta(t)$$

We introduce the fourier transform of $G(\mathbf{r},t)$ with respect to r :

$$G(\mathbf{r},t) = \frac{1}{(2\pi)^3}\int d^3\mathbf{k}\,\tilde{G}(\mathbf{k},t)\,e^{i\mathbf{k}.\mathbf{r}}$$

and write the delta function $\delta^3(\mathbf{r})$ by its usual formula

$$\delta(\mathbf{r}) = \frac{1}{(2\pi)^3}\int d^3\mathbf{k}\,e^{i\mathbf{k}.\mathbf{r}}$$

we find after substituting in the heat equation and comparing both sides

$$\frac{\partial\tilde{G}(\mathbf{k},t)}{\partial t} = -D\mathbf{k}^2\tilde{G}(\mathbf{k},t) + \delta(t)$$

which can be solved as follows :

Consider the equation

$$\left(\frac{d}{dt} + a\right)F = \delta(t)$$

We already know that $\theta'(t) = \delta(t)$. Therfore we try a solution of the type $F(t) = \theta(t)f(t)$ where f is an normal unknown function. Substituting we get

$$\delta(t)f(t) + \theta(t)f'(t) + a\theta(t)f(t) = \delta(t)f(0) + \theta(t)f'(t) + a\theta(t)f(t) = \delta(t)$$

This implies that $f(0) = 1$ and $f'(t) = -af(t)$ for $t > 0$. Therefore the solution is

$$F(t) = \theta(t)e^{-at}$$

Coming back :

$$\tilde{G}(\mathbf{k},t) = \theta(t)\exp[-D\mathbf{k}^2t]$$

therefore

$$G(\mathbf{r},t) = \frac{\theta(t)}{(2\pi)^3}\int d^3\mathbf{k}\,\exp[-D\mathbf{k}^2t]\,e^{i\mathbf{k}.\mathbf{r}}$$

We can perform the three dimensional integral on k by noting that there are actually three independent gaussian integrals

$$\begin{aligned}\int d^3\mathbf{k}\, \exp[-D\mathbf{k}^2 t]\, e^{i\mathbf{k}.\mathbf{r}} &= \int dk_x\, \exp[-Dk_x^2 t + ik_x x] \int dk_y\, \exp[-Dk_y^2 t + ik_y y] \\ &\times \int dk_z\, \exp[-Dk_z^2 t + ik_z z]\end{aligned}$$

The first integral gives for example

$$\int dk_x\, \exp[-Dk_x^2 t + ik_x x] = \sqrt{\frac{\pi}{Dt}}\exp[-x^2/(4Dt)]$$

and similarly for the other two factors.

The result is

$$G(\mathbf{r}, t) = \frac{\theta(t)}{(2\sqrt{\pi D t})^3}\, \exp[-\mathbf{r}^2/(4Dt)]$$